IJCSE International Journal of Computer Sciences and Engineering Open Access
Research Paper Vol.-7, Special Issue, 5, March 2019 E-ISSN: 2347-2693

Geodetic Variants of Sierpinski Triangles

Deepa Mathew[1]*, D. Antony Xavier[2]

Department of Mathematics, Loyola College Chennai,India

**Corresponding Author: deepamathew32@gmail.com*

DOI: https://doi.org/10.26438/ijcse/v7si5.96100 | Available online at: www.ijcseonline.org

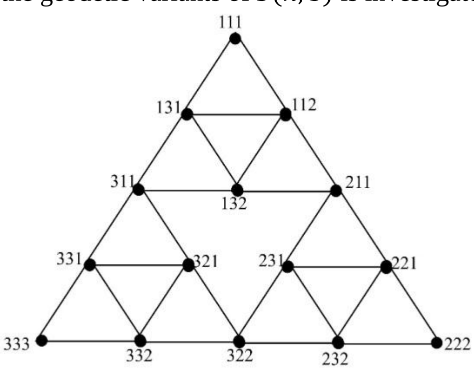
Abstract— The concept of convex sets in the classical Euclidean geometry was extended to graphs and different graph convexities were studied based on the kind of path that is considered. The geodetic number of a graph is one of the extensively studied graph theoretic parameters concerning geodesic convexity in graphs. A $u-v$ geodesic is a $u-v$ path of length $d(u,v)$ in G. For a non-trivial connected graph G , a set $S \subseteq V(G)$ is called a geodetic set if every vertex not in S lies on a geodesic between two vertices from S. The cardinality of the minimum geodetic set of G is the geodetic number $g(G)$ of G. The Sierpinski triangle , also called the Sierpinski gasket or the Sierpinski Sieve, is a fractal and attractive fixed set with the overall shape of an equilateral triangle, subdivided recursively into smaller equilateral triangles. In this paper some of the geodetic variants including hull number, monophonic hull number, geodetic number, strong geodetic number , total geodetic number ,upper geodetic number, open geodetic number and strong open geodetic number for Sierpinski triangle is investigated.

***Keywords* —** geodetic number, strong geodetic number, total geodetic number, hull number**.**

I. INTRODUCTION

Let $G=(V(G),E(G))$ be a connected graphs without loops and multiple edges and let the order of G be $|V(G)|=n$. The distance $d(u,v)$ is a length of the shortest $u-v$ path in$G$. An $u-v$ geodesic is an $u-v$ path of length $d(u,v)$ in G. A vertex x is said to lie on an $u-v$ geodesic P if x is an internal vertex of P. The eccentricity $e(u)$ of a vertex u is defined by $e(u) = max\{d(u,v\} : v \in v\}$. The minimum and the maximum eccentricity among vertices of G is its radius r and diameter d, respectively. For graph theoretic notation and terminology, we follow [2]. Harary et al introduced a graph theoretical parameter in [7] called the geodetic number of a graph and it was further studied in [8]. In [7] the geodetic number of a graph is as follows, let $I(u,v)$ be the set of all vertices lying on some $u-v$ geodesic of , and for some no empty subset S of $V(G), I(S) = \cup_{u,v\in S} I(u,v)$. The set S of vertices of G is called a geodetic set of G .If $I(S) = V$ and a geodetic set of minimum cardinality is called minimum geodetic set of G. The cardinality of the minimum geodetic set of G is the geodetic number $g(G)$ of G. The problem of finding geodetic number of a graph is shown to be an NP-hard problem [14]. In [4] the geodetic number is also referred as geodomination number. Chartrand, Harary Swart and Zhang were the first to study the geodetic concepts in relation to domination. Strong geodetic is a variation of geodetic concept which finds its application in social networks. Let G be a graph. Given a set $S \subseteq V(G)$, for each pair of vertices (x,y) ⊆ S, $x \neq y$, let $\tilde{g}(x,y)$ be a selected fixed shortest path between x and y.Let $\tilde{I}(S) = \{\tilde{g}(x,y) : x,y \in S\}$ and $(\tilde{I}(S)) = \cup_{\tilde{P}\in\tilde{I}(S)} V(\tilde{P})$. If $V\left(\tilde{I}(S)\right) = V$ for some $\tilde{I}(S)$, then S is called a strong geodetic set.[13] The strong geodetic problem is find a minimum strong geodetic set S of G. The cardinality of the minimum strong geodetic set is the strong geodetic number of G and is denoted by $sg(G)$. A geodetic set $S \subseteq V(G)$ is a total geodetic set if the subgraph $G[S]$ induced by S has no isolated vertices. The minimum cardinality of a total geodetic set is the total geodetic number $g_t(G)$. The total geodetic number is only defined for graphs with no isolated vertices.[1] The convex hull of a set S of vertices of a graph G is the smallest set T containing S such that all vertices on any geodesic joining vertices of T lie in T and is denoted as $[S]$. The set S is said to be convex if $[S]=S$. The number of vertices in a smallest subset H of S such that $[H]=S$ is called the hull number of S denoted $h(S)$.[6] For any two vertices u and v in a connected graph G, a $u-v$ path is a monophonic path if it contains no chords, and the monophonic distance $dm(u,v)$ from u to v is defined as the length of a longest $u-v$ monophonic path in G. A $u-v$ monophonic path of length $dm(u,v)$ is called a $u-v$ monophonic.[12] For two vertices u and v, let $J[u,v]$ denotes the set of all vertices which lie on $u-v$ monophonic path. For a set S of vertices, let ∪u,v∈S$J(u,v)$ = J(S) .The set S is monophonic convex or m — convex if $J[S] = S$.The smallest monophonic convex set containing S is denoted by $J_h[S]$ and called the monophonic convex hull of S.A subset $S \subseteq V(G)$ is called a monophonic

set if $J(S) = V(G)$ and a monophonic hull set if $J_h[S] = V(G)$.[12] A set S of vertices in a connected graph G is an open geodetic set if for each vertex v in G, either (1) v is an extreme vertex of G and $v \in S$ or (2) v is an internal vertex of an $x - y$ geodesic for some $x, y \in S$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the open geodetic number $og(G)$ of G. [18]A strong geodetic set S of vertices in a connected graph G is an strong open geodetic set if for each vertex v in G, either (1) v is an extreme vertex of G and $v \in S$ or (2) v is an internal vertex of an $x - y$ geodesic for some fixed $x, y \in S$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the open geodetic number $sog(G)$ of G.[19]
The generalized Sierpinski graph $S(n, k)$ is a fractal. The Sierpinski gaskest graph $S[n, k]$ is constructed from $S(n, k)$ by contracting all the edges that does not lie in K_k.[17] Sierpinski-like graphs $S^{+}(n, k)$ and $S^{++}(n, k)$ were introduced as an attempt to make the Sierpinski graphs regular. [10] In this paper the value of k is restricted to 3 and some of the geodetic variants of $S(n, 3)$ is investigated.



S[3,3]

II. MAIN RESULTS

Theorem 1:- For a Sierpinski graph $S(n, 3)$ the hull number is 3

Proof:- This can be proved by induction on n.

It is clear the the $h(S(2,3) = 3$. Assume the result is true up to n

Consider $(n + 1,3)$. $S(n + 1,3)$ consists of 3 copies of S(n,3) namely $S_1(n, 3), S_2(n, 3), S_3(n, 3)$ respectively and assume $\{\alpha_1, \beta_1, \gamma_{1\}}\}$, $\{\alpha_2, \beta_2, \gamma_2\}$, $\{\alpha_3, \beta_3, \gamma_3\}$ are the set of simplicial vertices in the each of these three copies.

Let $\alpha = 000 \dots 0(n\ times)$, $\beta = 111 \dots .1(n\ times)$, $\gamma = 222 \dots 2(n\ times)$ with $S = \{\alpha, \beta, \gamma\}$ .Then $\{\alpha, \gamma_1, \gamma_2, \beta_2, \alpha_3, \gamma_3, \beta_{1,}, \alpha_2, \beta_{3,}\} \subseteq I(S)$.

Also by induction hypothesis

$$H(\alpha, \beta_1, \gamma_1) = V[S_1(n, 3)]0,$$
$$H(\beta, \gamma_2, \alpha_2) = V[S_2(n, 3)] \quad and$$
$$H(\gamma, \alpha_3, \beta_3) = V[S_3(n, 3)].$$

Since $S_1(n, 3), S_2(n, 3),$ and $S_3(n, 3)$ are the spanning sub graphs of $S(n + 1,3)$. Therefore $H[\alpha, \beta, \gamma] = V[S(n + 1,3)]$.

Theorem 2:- For a Sierpinski graph $S(n, 3)$, the monophonic hull number is 3

Proof:- Each extreme vertex of $S(n, 3)$ would belong to its every monophonic hull set. But for $S(n, 3)$ there exists exactly 3 extreme vertices. Hence the monophonic hull set, $mh(S(n, 3) \geq 3$. Also for any connected graph $G, mh(G) \leq h(G)$. From Theorem:1 $h(S(n, 3) = 3$,which implies $mh(S(n, 3) \leq 3$. Thus $mh(S(n, 3) = 3$

Theorem 3:- For a Sierpinski graph $S(n, 3) \geq 2$,the geodetic number is given by $g\big(S(n, 3)\big) = 2 * 3^{n-3} + 3$

Proof:- Let $S(n, 3)$ be the Sierpinski graph. Consider any cycle C_n of length 12. Partition the vertices of C_{12} into two sets X and Y as shown in the figure (Ref fig 1), where $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5, y_6\}$.The three simplicial vertices would belong to $g(S(n, 3))$. It is evident that for any vertex $u, v \in V(S(n, 3))/N(V(C_{12}))$,the $u - v$ geodesic will not traverse through the vertices of set X. If $u, v \in N(V(C_{12}))$, then atleast three vertices are required to geodominate the vertices of set X. But any two antipodal vertices in C_{12} would geodominate all the vertices of set X. Since there is a total of 3^{n-3} cycles of length 12 in $S(n, 3)$,the geodetic number is $\big(S(n, 3)\big) \geq 2 * 3^{n-3} + 3$.
Also we can construct a simple algorithm consisting of any two antipodal vertices from the set X of all possible C_{12}(there are 3^{n-3} C_{12} cycles) and the simplicial vertices which forms a geodetic set. Therefore $g\big(S(n, 3)\big) = 2 * 3^{n-3} + 3$.

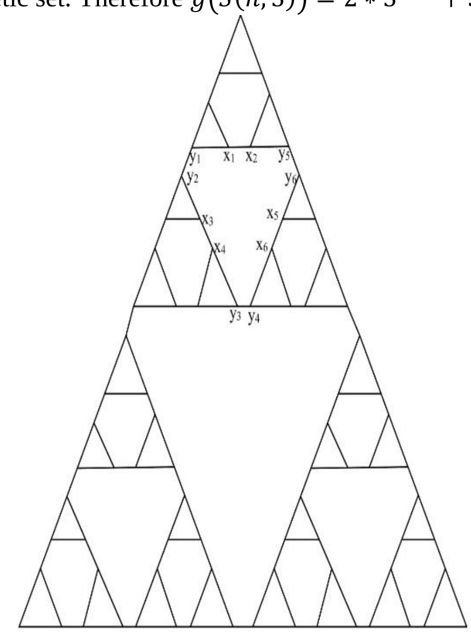


Fig 1: S(4,3)

Theorem 4:- For a Sierpinski graph $S(n,3) \geq 2$,the total geodetic number is given by $g_t(S(n,3)) \geq 4 * 3^{n-3} + 6$

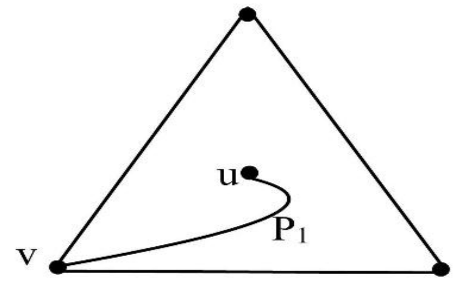
Proof: - Consider the Sierpinski graph $S(n,3)$. The set of simplicial vertices along with the stem vertices would belong to the total geodetic set of $S(n,3)$.The geodetic set of $S(n,3)$ contains any two antipodal vertices from each of its 3^{n-3} cycles of length 12. But for any $n \geq 4$ the $g_t(C_n)$ =4. Hence alongwith each pair of the antipodal vertices considered from each cycle of lengh 12 in $S(n,3)$, one of its adjacent vertex should also be considered to total geodominate $S(n,3)$.Thus $g_t(S(n,3)) \geq 4 * 3^{n-3} + 6$

Theorem 5:- For a Sierpinski graph $S(n,3) \geq 2$, the strong geodetic number is given by $sg(S(n,3)) \geq 2 * 3^{n-3} + 3$

In [5] it is given that the $g(G) \leq sg(G)$. Therefore by Theorem 3, $sg(S(n,3)) \geq 2 * 3^{n-3} + 3$.

Theorem 6:- For a Sierpinski graph $S(n,3) \geq 2$, the strong geodetic number is given by $sg(S(n,3)) = 2 * 3^{n-3} + 3$

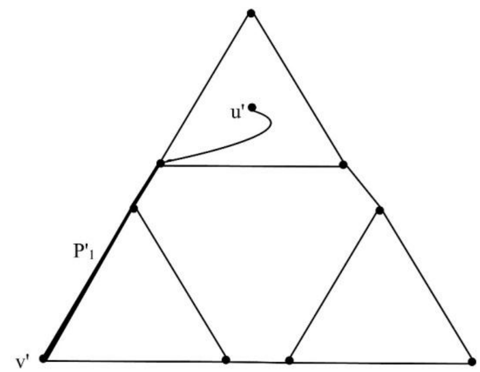
Proof: - By the previous theorem $(S(n,3)) \geq 2 * 3^{n-3} + 3$. It is left to prove that the $sg(S(n,3)) \leq 2 * 3^{n-3} + 3$.
We use the induction hypothesis with the induction on n. It can be easily verified for $n = 2$ or 3. Assume that the theorem holds good for $S(n-1,3)$, that is $sg(S(n-1,3)) \leq 2 * 3^{n-4} + 3$. Let $S = S_1 \cup S_2$ be the minimum strong geodetic set for $S(n-1,3)$ *where* S_1 is a subset of internal vertices of $S(n-1,3)$ that contribute to the strong geodetic set with $|S_1| \leq 2.3^{n-4}$ *and* let S_2 be the set of all simplicial vertices of $S(n-1,3)$ with $|S_2| = 3$. Consider $v \in S_2$ and $u \in S_1$ with P_1 as a unique path between u and v. Refer figure 2: given below.



S(n-1,3)
Fig 2:

Let $S' = S_1' \cup S_2'$ where S_2' be the set of internal vertices of the minimum strong geodetic set from the three copies of $S(n-1,3)$ in $S(n,3)$ and S_1' as the set of all simplicial vertices of $S(n,3)$.For each P_1 in

$S(n-1,3)$ construct $P_1' = P_1 \cup \{u, v'\}$ as shown in fig 3:. Clearly there exists a $1-1$ correspondence between the set of paths of type P_1 in $S(n-1,3)$ and set of paths of type P_1' in S(n,3). S' is a strong geodetic set of S(n,3). Also $|S_2'| \leq 3 * 2 * 3^{n-4} = 2 * 3^{n-3}$ and $|S_1'| = 3$, which implies $|S'| \leq 2 * 3^{n-3} + 3$. This implies $sg(S(n,3)) \leq 2 * 3^{n-3} + 3$. Therefore $sg(S(n,3)) = 2 * 3^{n-3} + 3$.



S(n,3)

FIG 3:

Theorem 7:- For a Sierpinski graph $S(n,3)$ *with* $n \geq 2$,the upper geodetic number, $g^+(S(n,3)) = 3^{n-2} + 3$

Proof:- It is easy to verify that three vertices (which are equidistant) from each of the C_{12} cycles along with the simplicial vertices form an upper geodetic set . Therefore, $g^+(S(n,3)) \geq 3^{n-2} + 3$.
In $S(n,3)$, if one of the vertex is a simplicial vertex then each $S(2,3)$ contains two internal vertices of an upper geodetic set of $S(n,3)$. Otherwise each $S(2,3)$ contains exactly one vertex of any upper geodetic set of $S(n,3)$. Assume the contrary that one $S(2,3)$ contains two vertices x, y of an upper geodetic set
Case 1: As given in fig 4:, there would exists a simplicial vertex z and a shortest path P between z and one of x, y. Let P be the geodesic between z and also y lies on P ,which is a contradiction.
Case 2: Suppose x, y are the vertices of $S(2,3)$ and the previous construction is not possible.(Refer fig 5). Then there exists a vertex u from the neighbouring $S(2,3)$ and a simplicial vertex z such that y lies on the $u - z$ geodesic, which is a contradiction. This implies each $S(2,3)$ contains

exactly one element of any upper geodetic set. Therefore $g^{+}\left(S(n,3)\right) \leq 3^{n-2}+3$.Thus $g^{+}\left(S(n,3)\right) = 3^{n-2}+3$

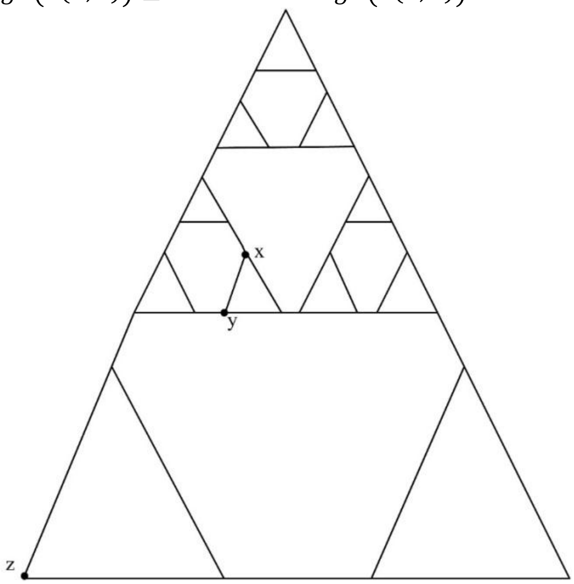


Fig 4:

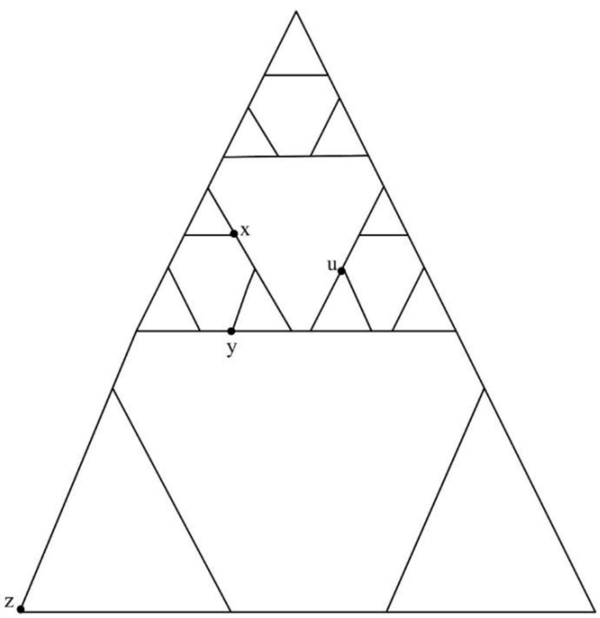


Fig 5:

Theorem 8:- For a Sierpinski graph $S(n,3)$ $with\ n \geq 2$,the open geodetic number, $og\left(S(n,3)\right) \leq 3^{n-2}+3$

Proof: Every open geodetic set contains all the extreme vertices of G. For any $(n,3)$, let $A=\{1,2,3\}$ be the set of its extreme vertices and $B=\{a,b,c,d,e,f,g,h,i,j,k,l\}$ be the vertex set of any C_{12} among its 3^{n-3} cycles of length 12 (Ref fig 6). From Theorem 3 the minimum geodetic set of $S(n,3)$ contains a pair of antipodal vertices from every C_{12} of $S(n,3)$**.** Choose a vertex from B that lies in the geodesic between any of the vertices in A. The vertices $\{a,g\} \in B$ forms a pair of antipodal vertices in C_{12} where b is an internal vertex of C_{12} and the vertex $\{a\}$ lies in the geodesic between the vertices$\{1,2\} \subseteq A$. But the vertex $\{g\}$ is not geodominated as it doesn't lie in any geodesic within or between the vertices of sets A and B. Hence to geodominate $\{g\}$, an adjacent vertex of it which is also an internal vertex of C_{12} should be chosen. Thus the minimum open geodetic set of $S(n,3)$ contains at least 3 vertices from each of its cycles of length 12 along with the 3 extreme vertices ,which implies $og\left(S(n,3)\right) \leq 3^{n-2}+3$.

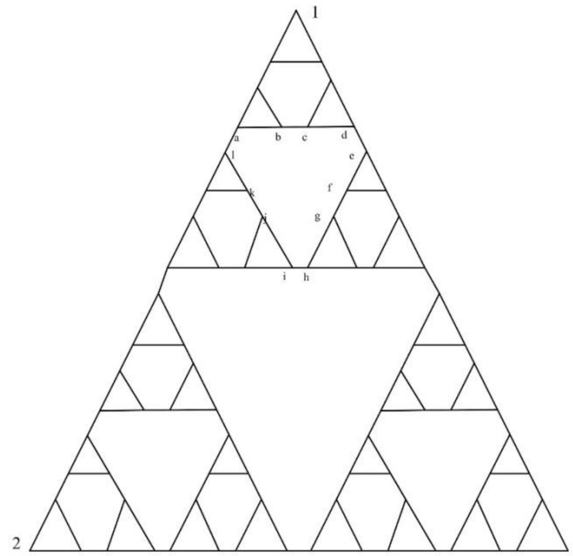


Fig 6:

Remark:- For a Sierpinski graph $S(n,3)$ $with\ n \geq 2$, the strong open geodetic number, $sog\left(S(n,3)\right) \leq 3^{n-2}+3$

For Sierpinski graph $S(n,3)$, the geodetic set itself forms a strong geodetic set. Similarly the open geodetic set itself forms a strong open geodetic set . Let T be an open geodetic set for S(n,3), then there exists ${}^{|T|}C_2$ geodesics in it and every vertex of $S(n,3)$ lies in a unique geodesic between the vertices of T.

REFERENCES

[1] Ahangar, H. Abdollahzadeh, and Vladimir Samodivkin. "The total geodetic number of a graph." Util. Math 100 (2016): 253-268.
[2] Buckley F., Harary F., "Distance in Graphs," Addison-Wesley, Redwood City, CA, 1990.
[3] Chartrand, Gary, Frank Harary, and Ping Zhang. "On the geodetic number of a graph." Networks39.1 (2002): 1-6.
[4] Chartrand, Gary, et al. "Geodominationin graphs." Bulletin of the Institute of Combinatorics and its Applications 31 (2001): 51-59.
[5] Dourado, Mitre C., et al. "Some remarks on the geodetic number of a graph." Discrete Mathematics 310.4 (2010): 832-837.
[6] Everett, Martin G., and Stephen B. Seidman. "The hull number of a graph." Discrete Mathematics 57.3 (1985): 217-223.

[7] Harary, Frank, Emmanuel Loukakis, and Constantine Tsouros. "The geodetic number of a graph." Mathematical and Computer
[8] Hansberg, Adriana, Lutz Volkmann, "On the geodetic and geodetic domination numbers of a graph," Discrete Mathematics, vol.310, no. 15, pp.2140-2146, 2010.
[9] Iršič, Vesna. "Strong geodetic number of complete bipartite graphs and of graphs with specified diameter." Graphs and Combinatorics 34.3 (2018): 443-456.
[10] Jakovac, Marko, and Sandi Klavžar. "Vertex-, edge-, and total-colorings of Sierpiński-like graphs." Discrete Mathematics309.6 (2009): 1548-1556.
[11] Jakovac, Marko. "A 2-parametric generalization of Sierpinski gasket graphs." Ars Comb. 116 (2014): 395-405.
[12] John, J., and V. Mary Gleeta. "The Forcing Monophonic Hull Number of a Graph." International Journal of Mathematics Trends and Technology 3.2 (2012): 43-46.
[13] Manuel, Paul, et al. "Strong geodetic problem in networks: computational complexity and solution for Apollonian networks." arXiv preprint arXiv:1708.03868 (2017).
[14] Pelayo, Ignacio M., Geodesic convexity in graphs, Springer, New York, 2013.Modelling 17.11 (1993): 89-95.
[15] Parisse, Daniele. "Sierpinski Graphs S (n, k)." Ars Combinatoria90 (2009): 145-160.
[16] Rodríguez-Velázquez, Juan Alberto, Erick David Rodríguez-Bazan, and Alejandro Estrada-Moreno. "On generalized Sierpiński graphs." Discussiones Mathematicae Graph Theory37.3 (2017): 547-560.
[17] Scorer, R. S., P. M. Grundy, and C. A. B. Smith. "Some binary games." The Mathematical Gazette 28.280 (1944): 96-103.
[18] A. P. Santhakumaran and T. Kumari Latha, On the open geodetic number of a graph, SCIENTIA, Series A :Mathematical Sciences, 19 (2010),131-142.
[19] Xavier, D. Antony, S. ARUL AMIRTHA RAJA"Strong Open geodetic Set in Graphs"(Communicated)