IJCSE International Journal of Computer Sciences and Engineering Open Access
Research Paper Volume-4, Special Issue-6, Aug 2016 E-ISSN: 2347-2693

Volterra Integral Equations via Triangular and Hybrid Orthogonal Functions

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Available online at: www.ijcseonline.org

Received: Jun/26/2016 Revised: July/08/2016 Accepted: July/22/2016 Published: Aug/12/2016

Abstract— We have expounded a new simple algorithm to solve non-linear Volterra integral equations via HF and drawn a comparative study between HF and TF in solving two classes of Volterra integral equations, i.e. Volterra integral equation of 2nd kind and Volterra-Hammerstein equation. To exemplify the usability of this algorithm we have dealt an optimal control problem of a deterministic system via this algorithm.

Keywords— Non-linear Volterra Integral Equations of 2nd Kind, Non-linear Volterra-Hammerstein Equations, Hybrid Functions, Triangular Functions, Optimal control, Cost function.

I. Introduction

Non-linear Volterra integral equations play key role in different scientific fields [1-2], such as potential concept and Dirichlet difficulties, electrostatics, the particle transference difficulties of astrophysics and reactor theory, diffusion difficulties, heat transfer difficulties, people dynamics, spread of epidemics, and semi-conductor devices.

In this present paper we have done a comparative study on solution of non-linear Volterra integral equations by orthogonal Hybrid function (HF) [3], a combination of SHF (sample and hold function) and RHTF (right hand triangular function) and Triangular function (TF) [4], a combination of LHTF (left hand triangular function) and RHTF.

Here we have concentrated on the following classes of non-linear Volterra integral equations-

1. Non-linear Volterra integral equations of 2nd kind
2. Non-linear Volterra-Hammerstein equations

In past, several numerical methods were developed to approximate solutions of previously stated classes of Volterra integral equations. Sepehrian and Razzaghi presented a single-term Walsh series method to solve non-linear Volterra-Hammerstein integral equations in [5] where as Maleknejad et al [6] and Mandal and Bhattacharya [7] both have dealt Volterra integral equations with Bernstein polynomials. The single-term Walsh series method we will obtain piece wise constant values whereas this approach yields values at sample points. As a whole we can say Hybrid function has been generated from Trapezoidal method with some restrictions.

Here, we have approximated each function in HF and TF domain and eventually reduced it to a system of non-linear equations, can be easily solved by Newton-Raphson method.

II. Brief Discussion On Volterra Integral Equations

The afore-stated classes of Volterra integral equations have the following generalized forms, e.g.

1. Non-linear Volterra integral equations of 2nd kind:

$$g(t) = f(t) + \int_0^t K(t,s)\phi(g(s))ds \qquad (1)$$

Non-linear Volterra Hammerstein integral equation:

$$g(t) = f(t) + \int_0^t K(t,s)\phi(s, g(s))ds \qquad (2)$$

Here, $g(t)$ is the unknown function. $f(t)$ and kernel $K(t,s)$ are real valued function. $\phi(g(s))$, $\phi(s, g(s))$ are the non-linear function of $g(s)$.

III. Brief Discussion On Triangular Orthogonal Function (TF)

A. Deb et al. presented a new class of orthogonal function in [4], i.e. triangular function, which is composed of right hand triangular function and left hand triangular function set.

A. Definition

Right hand triangular function (RHTF) and left hand triangular function (LHTF) are defined as,

$$T1_i(t) = 1 - ((t - ih)/h) \qquad ih \le t < (i+1)h$$
$$= 0 \qquad otherwise \qquad (3.a)$$

$$T2_i(t) = (t - ih)/h \qquad ih \le t < (i+1)h$$
$$= 0 \qquad otherwise \qquad (3.b)$$

where, $T1_i(t)$ and $T2_i(t)$ are the *(i+1)*th term of $T1_{(m)}(t)$ and $T2_{(m)}(t)$ m- vector set respectively.

$$T1_{(m)}(t) = [T1_0(t), T1_1(t), ...T1_{m-1}(t)]^T$$
$$T2_{(m)}(t) = [T2_0(t), T2_1(t), ...T2_{m-1}(t)]^T \qquad (4)$$

where T = length of the interval considered, m = no. of samples or no. of members in $T1_{(m)}(t)$ vector or $T2_{(m)}(t)$ vector, h = sample spacing $= T/m$, $i = 0,1,2,.........,m-1$.

B. Orthogonality of TF

Orthogonal property of 1D-TF is, $\int_0^T T1_i^p T2_j^q = \Delta_{p,q}\delta_{i,j}$. $\delta_{i,j}$ is Kronecker delta function, which yields 1 for $i = j$ and 0 for $i \ne j$ and,

$$\Delta_{p,q} = h/2, \qquad p = q \in [1,2]$$

C. Function Approximation by TF

A square integrable function $y(t)$ of Lebesgue measure can be approximated in TF domain as follows,

$$y(t) \approx [a_0, a_1, ...a_{m-1}]T1_{(m)}(t) + [b_0, b_1, ...b_{m-1}]T2_{(m)}(t) \square Y1^T T1(t) + Y2^T T2(t)$$
$$= \begin{pmatrix} Y1^T & Y2^T \end{pmatrix} \begin{pmatrix} T1(t) \\ T2(t) \end{pmatrix} = Y^T T(t) \qquad (5)$$

where, $a_i = f(ih) = f(t_i)$ and $d_i = c_{i+1} = f((i+1)h) = f(t_{i+1})$.

D. Operational Matrices for Integration

Operational matrices for integration in TF domain is [4],

$$[\int_0^t T1_{(m)}(s)ds] = (h/2)[R1_{(m\times 2m)}]T_{(2m)}(t) \qquad (6.a)$$

$$[\int_0^t T2_{(m)}(s)ds] = (h/2)[R1_{(m\times 2m)}]T_{(2m)}(t) \qquad (6.b)$$

$$[R1_{(m\times 2m)}] = [\square 0,1,1,..........1,1\square_{(m\times m)} \vdots \square 1,1,1,..........1,1\square_{(m\times m)} \qquad (6.c)$$

Hence,

$$\int_0^t y(s)ds = Y1^T \int_0^t T1_{(m)}(s)ds + Y2^T \int_0^t T2_{(m)}(s)ds$$
$$= (h/2)(Y1^T + Y2^T)R1_{(m\times 2m)}T_{(2m)}(t) \qquad (7)$$

IV. A BRIEF REVIEW ON HYBRID ORTHOGONAL FUNCTION (HF)

In 2012, A. Deb et al. proposed a new set of orthogonal function, namely hybrid function in [3] which is a combination of sample-and-hold function and right hand triangular function set.

A. Definition

An m-set one dimensional hybrid function (1D-HFs) consists of an m-set SHF ($H1_{(m)} = S_{(m)}$) and an m-set RHTF ($H2_{(m)} = T2_{(m)}$).

We express $H1_{(m)}(t)$ and $H2_{(m)}(t)$ as follows,

$$H1_{(m)}(t) = [H1_0, H1_1, ..., H1_{m-1}]^T \qquad (8.a)$$

$$H2_{(m)}(t) = [H2_0, H2_1, ..., H2_{m-1}]^T \qquad (8.b)$$

where T = length of the interval considered, m = no. of samples or no. of members in $H1_{(m)}(t)$ vector or $H2_{(m)}(t)$ vector, h = sample spacing = T/m.

*(i+1)*th terms of $H1_{(m)}(t)$ and $H2_{(m)}(t)$ are defined as,

$$H1_i(t) = 1, \qquad ih \le t < (i+1)h$$
$$= 0, \qquad otherwise \qquad (9.a)$$

$$H2_i(t) = (t - ih)/h, \quad ih \le t < (i+1)h$$
$$= 0, \qquad otherwise \qquad (9.b)$$

where $i = 0,1,2,.........,m-1$.

B. Orthogonality of HF

Orthogonality of 1D-HF is, $\int_0^T H1_i^p H2_j^q = \Delta_{p,q}\delta_{i,j}$.

$\delta_{i,j}$ is Kronecker delta function, which yields 1 for $i = j$ and 0 for $i \neq j$ and,

$$\Delta_{p,q} = h, \quad p = q = 1$$
$$= h/3, \quad p = q = 2$$

C. Function Approximation by HF

A square integrable function $f(t)$ of Lebesgue measure is expanded into an m-set HF series in $t \in [0, \mathrm{T})$ [3],

$$f(t) \approx [c_0, c_1, ...c_{m-1}]H1_{(m)}(t) + [d_0, d_1, ...d_{m-1}]H2_{(m)}(t) \triangleq F1^T H1(t) + F2^T H2(t)$$

$$= \begin{pmatrix} F1^T & F2^T \end{pmatrix} \begin{pmatrix} H1(t) \\ H2(t) \end{pmatrix} = F^T H(t) \tag{10}$$

where, $c_i = f(ih) = f(t_i)$ and

$d_i = c_{i+1} - c_i = f((i+1)h) - f(ih) = f(t_{i+1}) - f(t_i)$.

D. Operational Matrix for Integration in HF Domain

As stated in [3], the operational matrices for integration in HF domain are:

$$[\int_0^t H1_{(m)}(s)ds] = h[R2_{(m\times 2m)}]H_{(2m)}(t)$$

$$[\int_0^t H2_{(m)}(s)ds] = (h/2)[R2_{(m\times 2m)}]H_{(2m)}(t) \tag{11}$$

$$[R2_{(m\times 2m)}] = [\lceil 0,1,1,...........1,1 \rfloor_{(m\times m)} \vdots I_{(m)}]$$

Hence,

$$\int_0^t f(s)ds = F1^T \int_0^t H1_{(m)}(s)ds + F2^T \int_0^t H2_{(m)}(s)ds$$

$$= h(F1^T + (1/2)F2^T)R2_{(m\times 2m)}H_{(2m)}(t) \tag{12}$$

V. SOLVING VOLTERRA INTEGRAL EQUATIONS VIA TF AND HF

In this section, we have presented an efficient method with minimal error to solve Volterra integral equations in TF and HF domain based on the property previously stated.

Volterra integral equation of 2nd kind and Volterra-Hammerstein integral equation have the following generalized form respectively from eq.(1) and (2)-

$$g(t) = f(t) + \int_0^t K(t,s)\phi(g(s))ds$$

$$g(t) = f(t) + \int_0^t K(t,s)\phi(s, g(s))ds$$

Here $g(t), f(t), K(t,s), \phi(g(s)), \phi(s, g(s))$ are real-valued, square integrable functions of Lebesgue measure. $g(t)$ is the function to be approximated and $\phi(g(s)), \phi(s, g(s))$ are nonlinear functions of $g(s)$.

Here we are representing piece-wise linear basis function TF ($T(t)$) or HF ($H(t)$) by $M(t)$.Depending on $M(t) = T(t)$ or $M(t) = H(t)$, only the coefficient matrices will change, the rest of the procedure will be same for both the cases.

$$g(t) = M^T_{(2m)}(t)G_{(2m\times 1)}$$

$$f(t) = M^T_{(2m)}(t)F_{(2m\times 1)}$$

$$\begin{aligned} K(t,s) &= M1^T_{(m)}(t)(K11_{(m\times m)}M1_{(m)}(s) + K21_{(m\times m)}M2_{(m)}(s)) + \\ &\quad M2^T_{(m)}(t)(K12_{(m\times m)}M1_{(m)}(s) + K22_{(m\times m)}M2_{(m)}(s) \\ &= M^T_{(2m)}(t)K_{(2m\times 2m)}M_{(2m)}(s) \end{aligned} \tag{13}$$

$$M^T_{(2m)}(t) = \begin{pmatrix} M1^T_{(m)}(t) & M2^T_{(m)}(t) \end{pmatrix}_{(1\times 2m)}$$

$$K_{(2m\times 2m)} = \begin{pmatrix} K11_{(m\times m)} & K21_{(m\times m)} \\ K12_{(m\times m)} & K22_{(m\times m)} \end{pmatrix}_{(2m\times 2m)}$$

$$\phi(g(s)) = M^T_{(2m)}(s)\Delta_{(2m\times 1)}$$

$$\phi(s, g(s)) = M^T_{(2m)}(s)\Psi_{(2m\times 1)}$$

To solve eq. (1) we require the following lemma,

Lemma 1: Let $M_{(2m)}(s)$ be the 2m piece-wise linear basis vector, then

$$M_{(2m)}(s)M^T_{(2m)}(s) = \begin{pmatrix} diag(M1_{(m)}(s))_{(m\times m)} & 0_{(m\times m)} \\ 0_{(m\times m)} & diag(M2_{(m)}(s))_{(m\times m)} \end{pmatrix}_{(m\times m)} \tag{14}$$

Now, to solve eq.(1), we put (13) into (1),

$$M^T_{(2m)}(t)G_{(2m\times1)}$$
$$= M^T_{(2m)}(t)F_{(2m\times1)} + \int_0^t M^T_{(2m)}(t)K_{(m\times m)}M_{(2m)}(s) M^T_{(2m)}(s)\Delta_{(2m\times1)}ds$$
$$= M^T_{(2m)}(t)F_{(2m\times1)} + M^T_{(2m)}(t)K_{(m\times m)} \int_0^t diag(M_{(2m)}(s))ds\Delta_{(2m\times1)}$$
$$= T^T_{(2m)}(t)F_{(2m\times1)} + T^T_{(2m)}(t)V_{(2m\times2m)}\Delta_{(2m\times1)}$$

where, if $M(t) = T(t)$, then

$$V_{(2m\times2m)} = (h/2)\begin{pmatrix} K11_{L\Delta} & K21_{L\Delta} \\ K12_{L\Delta}+K12_d & K22_{L\Delta}+K22_d \end{pmatrix}_{(2m\times2m)}$$

and, if $M(t) = H(t)$, then

$$V_{(2m\times2m)} = (h/2)\begin{pmatrix} K11_{L\Delta} & K21_{L\Delta} \\ K12_d & K22_d \end{pmatrix}_{(2m\times2m)}$$

In which, $K11_{L\Delta}$ is the lower triangular matrix of $K11$ and $K12_d$ is a diagonal matrix with the diagonal elements of $K12$.

The system of non-linear equations equivalent to generalized form of Volterra integral equation of 2nd kind is:

$$G_{(2m\times1)} = F_{(2m\times1)} + V_{(2m\times2m)}\Delta_{(2m\times1)} \qquad (15).$$

Following the same steps as above, we will obtain the system of non-linear equations equivalent to generalized form of Volterra-Hammerstein integral equation, i.e.

$$G_{(2m\times1)} = F_{(2m\times1)} + V_{(2m\times2m)}\Psi_{(2m\times1)} \qquad (16).$$

VI. NUMERICAL EXAMPLES

Example 1: Consider the following example of Volterra integral equation of 2nd kind [2]:

$$g(t) = 1 + t^2 - te^{t^2} + \int_0^t e^{t^2-s^2-1}e^{g(s)}ds \qquad (17)$$

where, the actual solution of $g(t) = 1 + t^2$. Table 1 shows the *L^2* norm and L^infty norm error for TF and HF.

TABLE 1. Error Table for eq.(17)

No. of Samples, m	L^2 norm error for TF	L^infty norm error for TF	L^2 norm error for HF	L^infty norm error for HF
m=8	0.00000000	0.00000000	0.00000000	0.00000000
m=16	0.00000000	0.00000000	0.00000000	0.00000000
m=32	0.00000000	0.00000000	0.00000000	0.00000000

A graphical comparison is made in figure 1 and 2, between actual and approximated function by HF and TF for *m*=8, 16 and 32.

Here, L^2 norm and L^infty norm error are defined as follows, L^2 norm error:

$$|e| = |(g(t) - \hat{g}(t))| = \sqrt{\sum_{k=1}^{m} |(g(t) - \hat{g}(t))|^2} \text{ and}$$

L^infty norm error:

$$|e|_\infty = |(g(t) - \hat{g}(t))|_\infty = \max_i |(g(t) - \hat{g}(t))|.$$

where *m*=no. of samples, $e = m+1$ error vector, $g(t) = m+1$ coefficient vector of actual function, $\hat{g}(t) = m+1$ coefficient vector of approximated function via HF or TF.

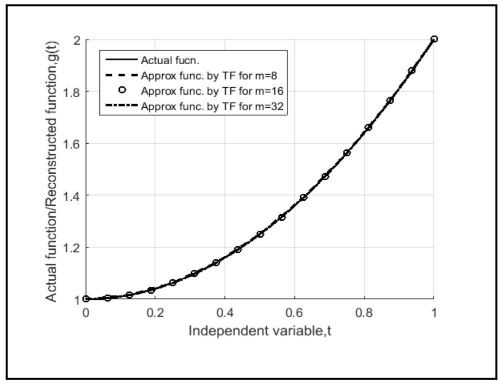

FIGURE 1. Actual function and reconstructed function by TF for m=8,16 and 32

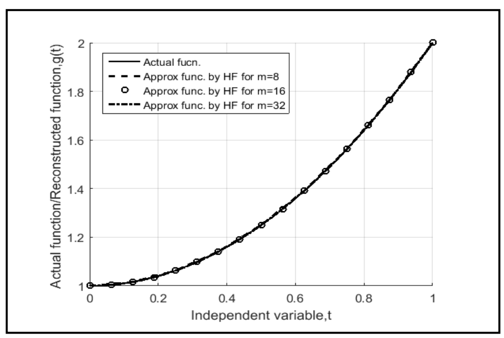

FIGURE 2. Actual function and reconstructed function by HF for m=8,16 and 32

Example 2: Consider the following example of a typical Volterra-Hammerstein integral equation [5]:

$$y(t) = 1 + \sin^2(t) - 3\int_0^t \sin(t-s)y^2(s)ds \qquad (18)$$

where, the actual solution of $y(t) = \cos(t)$. Table 2 shows the *L^2* norm and L^infty norm error for TF and HF. Figure 3 and 4 shows actual and approximated solution via TF and HF (for m=8, 16 and 32) of eq. (18) .

Table 2. Error Table for eq. (18)

No. of Samples, m	L^2 norm error for TF	L^infty norm error for TF	L^2 norm error for HF	L^infty norm error for HF
m=8	0.00113715	0.00055925	0.00113715	0.00055925
m=16	0.00039205	0.00014137	0.00039205	0.00014137
m=32	0.00013695	0.00003528	0.00013695	0.00003528

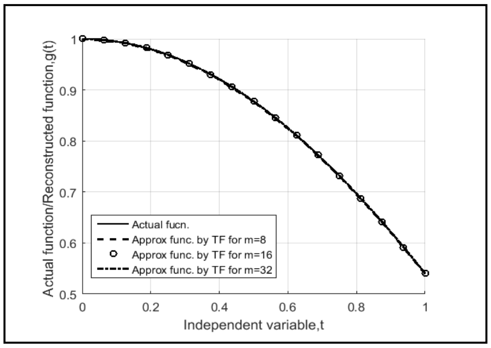

FIGURE 3. Actual function and reconstructed function by TF for m=8,16 and 32

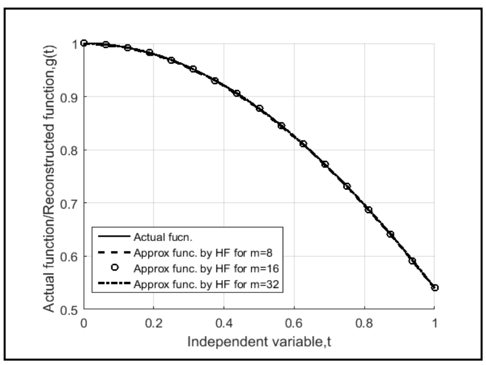

FIGURE 4. Actual function and reconstructed function by HF for m=8,16 and 32

VII. An Application: Volterra Class Of Equation In Optimal Control Of Deterministic System

In [8],[9],[10] optimal control law for deterministic system represented by integro-differential equation is studied via orthogonal functions such as block pulse function, shifted legendre polynomial etc.

We have developed an algorithm to determine the optimal control law and minimum cost function for a CSG by IDF via (PCLOBF) ,i.e., TF and HF.

Consider the following control system [8],[9],

$$x(t) = x(0) + \int_0^t x(\sigma) + u(\sigma) + \int_0^1 g(\sigma,\tau)\,x(\tau)\,d\tau\;\; d\sigma \qquad (19)$$

with,

$$g(\sigma,\tau) = 2 - 4(\sigma-\tau) \quad for\; 0 \le (\sigma-\tau) \le 0.5$$
$$= 0 \qquad for\; (\sigma-\tau) < 0 \;\; and\; (\sigma-\tau) > 0.5$$

and $x(0)$ =1.Cost function is stated as follows,

$$J = \frac{1}{2}\int_0^1 [x^2(t) + 2u^2(t)]\,dt \qquad (20)$$

Now we have determined $u(t)_{opt.}$, $x(t)$ for the above control system, shown in figure 5,6,7,8 and table 3 shows the value of $J_{\min}$ for m= 32, 64, 128, 256 in HF and TF approach.

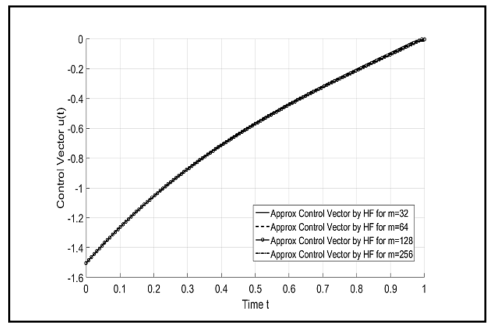

FIGURE 5. Estimated optimal control vector, $u(t)_{opt.}$, by HF approach for m=32, 64, 128, 256

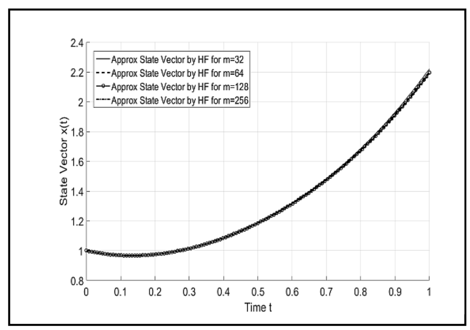

FIGURE 6. Estimated optimal state vector, $x(t)$, by HF approach for m=32, 64, 128, 256

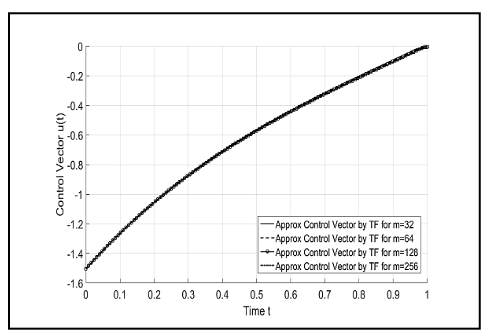

FIGURE 7. Estimated optimal control vector, $u(t)_{opt.}$, by TF approach for m=32, 64, 128, 256

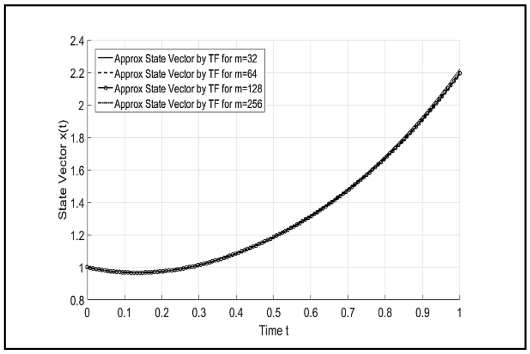

FIGURE 8. Estimated optimal state vector, $x(t)$, by HF approach for m=32, 64, 128, 256

Table 3. Cost function, J

No. of Samples, m	HF	TF
m=32	1.54808682	1.54808682
m=64	1.52535807	1.52535807
m=128	1.51423124	1.51423124
m=256	1.50872657	1.50872657

VIII. CONCLUSION

Here in this paper we have presented a much easier, straight-forward algorithm via HF and TF orthogonal basis functions. This method yields less computational burden than those method previously found in literature though it provides the same amount of accuracy which is obvious from the given examples. From the previously illustrated examples, it is evident that both TF and HF shows same amount of efficiency to solve Volterra integral equations.

ACKNOWLEDGMENT

We are truly grateful to Mr. Biswajit Paul of South Malda College for helping us throughout the work.

REFERENCES

[1] Abdul J. Jerri, "Introduction to Integral Equations with Applications", John Wiley & Sons Inc., 1999.

[2] Abdul-Majid Wazwaz, "Linear and Nonlinear Integral Equations Methods and Applications", Springer Heidelberg Dordrecht London New York, ISBN 978-3-642-21448-6, 2011

[3] A. Deb, G. Sarkar, A. Ganguly and A. Biswas, "Approximation, integration and differentiation of time functions using a set of orthogonal hybrid functions and their application to solution of first order differential equations", Applied Mathematics and Computation, Elsevier, 2012.

[4] A. Deb, A. Dasgupta, G. Sarkar, "A New Set of Orthogonal Functions and its Application to the Analysis of Dynamic Systems", Journal of Franklin Institute, 343 (1), page no.:1-26, (2006).

[5] B. Sepehrian and M. Razzaghi, "Solution of Nonlinear Volterra-Hammerstein Integral Equations via Single-Term Walsh Series Method", Mathematical Problems in Engineering, Hindawi Publishing Corporation, Volume (2005), Issue 5, pp 547-554, 2005.

[6] K. Maleknejad, E. Hashemizadeh and R. Ezzati, "A New Approach to the Numerical Solution of Volterra Integral Equations by Using Bernstein's Approximation", Commun. Nonlinear Sci. Numer. Simulat, 16, page no.:647-655, (2011).

[7] B.N. Mandal and Subhra Bhattacharya, "Numerical Solutions of Some Classes of Integral Equations Using Bernstein polynomials", *Appl. Mathe. Comput*, 190, page no.:1707-17016, (2007).

[8] C. Hwang, D.H. Shih and F.C. Kung , "Use of block-pulse functions in the optimal control of deterministic system", Int. J. Control, vol.44, no. 2, pp: 343-349,1986.

[9] D.H. Shih and F.C. Kung, "Optimal control of deterministic systems via shifted Legendre polynomials", IEEE trans. Automatic Control, vol. AC-31, no. 5, pp: 451-454,1986.

[1] J.H. Chou, "Application of Legendre series to the optimal control of integro-differential equations", Int J. Control, vol.45, no. 1, pp: 269-277,1987.

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