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Exact Wirelength of Embedding Locally Twisted Cube into Rooted Hypertree

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Abstract— The performance ability of a distributed multiprocessor is determined by its corresponding interconnection network and the primary criteria for choosing an appropriate interconnection network is its graph embedding capability. An embedding of a graph G into a graph H is an injective map on the vertices such that each edge of G is mapped into a shortest path of H. The wirelength of this embedding is the sum of the number of paths corresponding to G crossing every edge in H. In this paper we embed the locally twisted cube into rooted hypertrees to obtain the exact wirelength.

Keywords—Emedding, locally twisted cube, rooted hypertree, wirelength

I. INTRODUCTION

Interconnection networks play a major role in parallel processing and computing systems. An interconnection network can be represented in the form of a graph $G = (V(G), E(G))$ where the vertex set $V(G)$ represents the set of processors and the edge set $E(G)$ denote the set of communication links between different processors in the network. One of the most important performance factors of interconnection networks is how well other networks can be embedded into the particular network.

The graph embedding problem consists of a guest graph $G(V(G), E(G))$ and a host graph $H(V(H), E(H))$, both of which are connected, finite and undirected graphs on n nodes as shown in Fig.1 and is defined as follows. A Graph embedding [4,9] is an ordered pair $c$of injective maps where

i. f is a one-one map from $V(G)$ to $V(H)$.
ii. P_f is a one-to-one map from $E(G)$ to $\{P_f(u,v) : P_f(u,v)$ is a path in H between $f(u)$ and $f(v)$ for $(u,v) \in E(G)\}$.

One of the prime factors which determine a good quality embedding is the edge congestion. The edge congestion $EC_{\prec f,P_f \succ}(e)$ of an embedding is the maximum number of edges of the guest graph that are embedded on any single edge e of the host graph [6,11]. Explicitly,

$$EC_{\prec f,P_f \succ}(e) = |\{(u,v) \in E(G): e \in P_f(u,v)\}|$$

The wirelength [2,8] of an embedding $\prec f, P_f \succ$ from G into H is defined as

$$WL_{\prec f,P_f \succ}(G,H) = \sum_{e \in E(H)} EC_{\prec f,P_f \succ}(e)$$

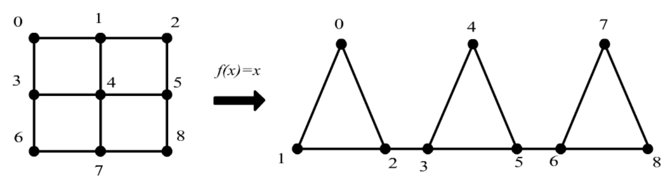

Figure 1: Embedding of G into H

The exact wirelength problem [10] is to find an embedding which induces $WL(G,H)$ where

$$WL(G,H) = \min_{\prec f,P_f \succ} WL_{\prec f,P_f \succ}(G,H).$$

Maximum Induced Subgraph Problem:

The maximum induced subgraph problem plays an important role in the wirelength problem. The problem is to find a subset of vertices of a given graph such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given m, if $I_G(m) =$

$max_{A\subseteq V,|A|=m}|I_G(A)|$ where $I_G(A) = \{(u,v) \in E : u,v \in A\}$, then the problem is to find a set $A \subseteq V$ such that $|A| = m$ and $I_G(m) = |I_G(A)|$. Such a set A is called an optimal set [7]. This optimal set helps in proving that the obtained wirelength is minimum.

Lemma 1. *(Congestion Lemma)* [11] Let G be an r-regular graph and $\prec f, P_f \succ$ be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S splits H into 2 components H_1 and H_2 and let $G_1 = G[f^{-1}(H_1)]$ and $G_2 = G[f^{-1}(H_2)]$. Also S satisfies the following conditions for $EC_{\prec f,P_f \succ}(S)$ to be minimum.

1. For every edge $(u,v) \in G_i, i = 1,2, P_f(u,v)$ has no edges in S.
2. For every edge (u,v) in G with $u \in G_1$ & $v \in G_2$, $P_f(u,v)$ has exactly one edge in S.
3. G_i, $i = 1,2$, is an optimal set.

Then $EC_{\prec f,P_f \succ}(S)$ is minimum and $EC_{\prec f,P_f \succ}(S) = r|V(G_1)| - 2|E(G_1)|$.

Lemma 2. [11] Let $\prec f, P_f \succ$ be an embedding from G into H. Let $E^k(H)$ be the set of edges with each edge repeated exactly k times. Let $\{S_1, S_2, \ldots, S_p\}$ be a partition of $E^k(H)$ such that $EC_{\prec f,P_f \succ}(S_i)$ is minimum for all i. Then $WL_{\prec f,P_f \succ}(G,H)$ is minimum and $WL_{\prec f,P_f \succ}(G,H) = \frac{1}{k}\sum_{i=1}^{p} EC_{\prec f,P_f \succ}(S_i)$.

The problem finds application in VLSI circuit design, task allocation to parallel computers such that the communication overhead could be reduced when the tasks are run in parallel and biological neural system designs. In the literature, the minimum wirelength problem when the host graph is a tree related structure has been discussed for hypercube, folded hypercube, enhanced hypercube and circulant network.

In this paper we obtain the exact wirelength of embedding a hypercube variant known as the locally twisted cube into a rooted hypertree. The rest of the paper is organized as follows. Section II gives some fundamental definitions and results related to the guest and host graph topologies. In Section III we use the edge partitioning method and compute the minimum wirelength of embedding locally twisted cubes into the rooted hypertree. We conclude the paper in Section IV.

II. Graph Topologies and Preliminaries

The locally twisted cube proposed by Yang et al. [15] keeps as many salient features of the binary hypercube as possible and is conceptually closer to the traditional hypercube, while it has diameter of about half of that of a hypercube of the same size. The n-dimensional locally twisted cube, LTQ_n, is constructed as follows.

Definition 1. [13] For $n \geq 2$, an n-dimensional locally twisted cube, LTQ_n, is a graph with vertex set of the form $\{0,1\}^n$. Two vertices $x = x_1 x_2 x_3 \ldots x_n$ and $y = y_1 y_2 y_3 \ldots y_n$ are adjacent if and only if either of the following conditions are satisfied.

1. There is an integer i with $1 \leq i \leq n-2$ such that $x_i = \bar{y}_i$ and $x_{i+1} = y_{i+1} \oplus x_n$, where $\oplus$ denotes the addition modulo 2, all the remaining bits of x and y being identical.
2. There is an integer $i \in \{n-1, n\}$ such that x and y differ only in the i^{th} bit.

The graph LTQ_4 is shown Fig.2

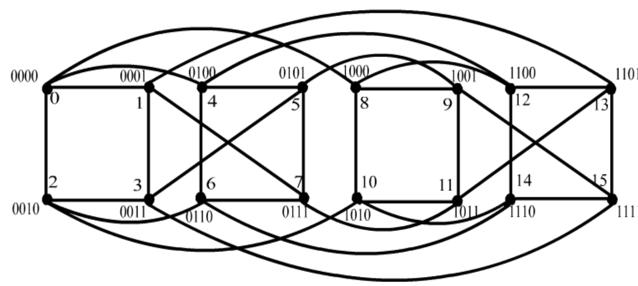

Figure 2: 4-dimensional locally twisted cube

For $1 \leq i \leq 2^{n-2}$, let $E_i = \{0,2,\ldots,2i-2\}$ and let $(TO)_i = \{a^t : 1 \leq t \leq i\}$ where $a_1 = 1, a_2 = 3$ and for $2 \leq k \leq n-1, 1 \leq j \leq 2^{k-1}$,

$$a_{2^{k-1}+j} = \begin{cases} a_j + 2^k + 2^k - 1 & : \ 1 \leq j \leq 2^{k-1} \\ a_j + 2^{k-1} & : \ 2^{k-2} < j < 2^{k-1} \end{cases}$$

Lemma 3. [3] For $1 \leq i \leq 2^n$,

$$(ETO)^i = \begin{cases} E_i & : \ 1 \leq i \leq 2^{n-1}{}_i \\ E_{2^{n-1}} \cup (TO)_{i-2^{n-1}} & : \ 2^{n-1} < i \leq 2^n. \end{cases}$$

is an optimal set in LTQ_n.

Definition 2. [12] Let T_n be a complete binary tree with n levels. The basic skeleton of a hypertree HT_n consists of two copies T_{n-1}^1 and T_{n-1}^2 of T_n and an additional set of edges induced between the corresponding vertices of the left and right subtrees. The rooted hypertree RHT_n is obtained from the hypertree HT_n by attaching a pendant vertex to its root vertex. The new vertex becomes the root of RHT_n, n ≥ 2.

Algorithm 1 Postorder Tree Traversal Algorithm

Do the following recursively until all nodes are traversed.
Step 1 - Traverse left subtree.
Step 2 - Traverse right subtree.
Step 3 - Visit root node.

We employ the following labeling and embedding algorithm for the minimum wirelength computation in Section III and show that it induces the minimum wirelength.

Table 1: Labeling and Embedding Algorithm

Guest Graph Labeling	**Host Graph Labeling**	**Embedding**
Label the vertices of LTQ_n by lexicographic ordering [5] from 0 to 2^{n-1}.	Label the vertices in the left complete binary tree of RHT_n by inorder labeling using $E = \{0,2,4,\dots,2^n-4,2^n-2\}$ and the vertices in the right complete binary tree by inorder labeling using $O = \{1,3,5,\dots,2^n-3,2^n-1\}$ such that the label 1 is adjacent to the label 0. The labeling pattern can be seen in Fig. 3	Define an embedding $\prec f, P_f \succ$ from LTQ_n into RHT_n by $f(x) = x$ together with $P_f(u,v)$ where $P_f(u,v)$ is a shortest path in the respective host graphs between $f(u)$ and $f(v)$ for $(u,v) \in E(LTQ_n)$

III. Main Results

In this section we embed the locally twisted cube LTQ_n into the rooted hypertree RHT_n to minimize the wirelength. The following result gives the base to prove that the embedding defined in Table 2 gives the minimum layout.

Lemma 4. [1] For $n \geq 3$, $j = 1,2,\dots,n-2$ and $i = 1,2,\dots,2^{n-j-1}$,

$$LT_i^j = \{2^{j+1}(i-1), 2^{j+1}(i-1)+1, 2^{j+1}(i-1)+2,\dots,2^{j+1}(i-1)+(2^{j+1}-3)\}$$ is an optimal set in LTQ_n.

Lemma 5. The embedding $\prec f, P_f \succ$ induces the minimum wirelength from LTQ_n into RHT_n.

Proof: The following table gives the edge cuts of RHT_n as shown in Fig. 3 and the vertex set of the components of RHT_n obtained from the removal of these edge cuts.

Table 2: Edge cuts of $E(RHT_n)$

Edge Cuts	**Components**	**V(Component)**
c_i^j $j = 1,2,\dots,n-2$ $i = 1,2,\dots,2^{n-j-1}$	$A_{ji}, \bar{A}_{ji}$	$V(A_{ji} = LT_i^j)$
E_c	B_1, B_2	$V(B_1) = \{0,2,4,..,2^n-2\}$
O_c	C_1, C_2	$V(C_1) = \{1,3,5,\dots,2^n-1\}$
EO_c	D_1, D_2	$V(D_1) = \{0,1,2,\dots,2^n-3\}$
S_0	F_1, F_2	$V(F_1) = \{2^n-1\}$

From Lemma 4, $f^{-1}\{V(A_{ji})\}$ and $f^{-1}\{V(F_1)\}$ are optimal in LTQ_n. The graphs $LTQ_n[f^{-1}\{V(B_1)\}]$, $LTQ_n[f^{-1}\{V(C_1)\}]$ and $LTQ_n[f^{-1}\{V(D_1)\}]$ are isomorphic to the induced subgraphs $LTQ_n[E_{2^{n-1}-1}]$, $LTQ_n[TO_{2^{n-1}-1}]$ and $LTQ_n[ETO_{2^{n-1}-2}]$ respectively. Hence by Lemma 3, the two graphs are optimal.

The edge cuts $\{c_i^j : j = 1,2,\dots,n-2, i = 1,2,\dots,2^{n-j-1}\} \cup E_c \cup O_c \cup EO_c \cup S_0$ satisfy conditions (i) - (iii) of Lemma 1, which implies the edge congestion of all the edges in these cuts are minimum. Let $e_i^j = c_i^j; j = 1,2,\dots,n-2, i = 1,2,\dots,2^{n-j-1}$ and $S_0 = SS_0$. Clearly, $\{c_i^j, e_i^j\} \cup E_c \cup O_c \cup EO_c \cup S_0 \cup SS_0$ form a partition of $E^2(H)$. Hence by Lemma 2, $\prec f, P_f \succ$ induces the minimum wirelength.

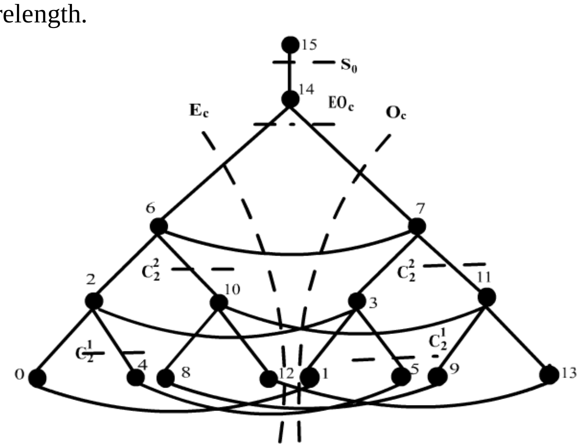

Fig 2: Edge cuts of RHT_4

Theorem 1: The minimum wirelength of embedding LTQ_n into RHT_n , $n \geq 2$ is given by $WL(LTQ_n, RHT_n) = 2^{n-1}(n^2 - 5n + 13) - (n + 7)$.

Proof: The following table gives the edge congestion of all the edge cuts of RHT_n given in Lemma 5.

Table 3

Edge Cuts	$EC_{<f,P_f>}$**(Edge Cuts)**
c_i^j, $j = 1,2,\dots,n-2$, $i = 1,2,\dots,2^{n-j-1}$	$EC_{<f,P_f>}(c_i^j) = 2\{(n-1)(2^j - 1) - 2j(2^{j-1} - 1)\}$
E_c	$EC_{<f,P_f>}(E_c) = n(2^{n-1} - 1) - 2(n-1)(2^{n-2} - 1)$
O_c	$EC_{<f,P_f>}(O_c) = n(2^{n-1} - 1) - 2(n-1)(2^{n-2} - 1)$
EO_c	$EC_{<f,P_f>}(EO_c) = 2(n-1)$
S_0	$EC_{<f,P_f>}(S_0) = n$

Therefore,

$$WL(LTQ_n, RHT_n) = \frac{1}{2}\left\{ 2\sum_{j=1}^{n-2}\sum_{i=1}^{n-j-1} EC_{<f,P_f>}(c_i^j) + EC_{<f,P_f>}(E_c) + EC_{<f,P_f>}(O_c) + EC_{<f,P_f>}(EO_c) + EC_{<f,P_f>}(S_0) \right\}$$
$$= 2^{n-1}(n^2 - 5n + 13) - (n + 7).$$

Fig. 4 shows the embedding of the 4-dimensional locally twisted cube into the hypertree of dimension 4 to optimize the wirelength.

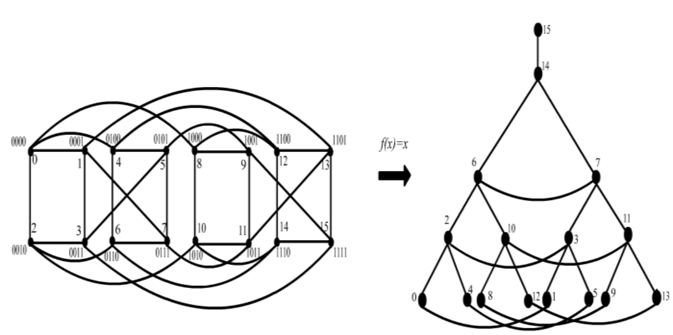

Fig 4. Embedding of LTQ_4 into RHT_4

The Matlab command for the minimum wirelength of locally twisted cube into rooted hypertree is given below.

Algorithm 2 Minimum wirelength computation

```
syms n
n = input('Enter the value of n:');
j = 3:n-1;
f = 2.^(n-j-1)*((2.*(2.^(j)-1).*n)-(2.*((j+1)).*2.^(j)-2.*j-1)));
X = sum(f);
L = 2*(n-1)+n+X;
display(L);
```

Algorithm Time Complexity: The n-dimensional locally twisted cube LTQ_n consists of $n = 2^n$ vertices. The assignment of labels to each vertex takes one unit of time, which means that n units of time are required for the assignment of $n-$ labels. In the same way RHT_n have $n = 2^n$ vertices each. Hence assignment of labels to the vertices of these graphs take another n units of time. Further, we assign one unit time for the $\frac{n}{2} + 1$ edge cuts and another $\frac{n}{2} - 1$ units of time for the minimum wirelength computation using edge partitioning technique.

Hence total time taken $= n + n + \frac{n}{2} + 1 + \frac{n}{2} - 1 \leq 3n$ for the wirelength computations, which shows that the algorithm time complexity is of order $O(n)$, which is linear.

IV. Conclusion

In this paper we have obtained the minimum wirelength of embedding the locally twisted cube into the rooted hypertree by solving the maximum induced subgraph problem for the locally twisted cube using edge partitioning technique and Congestion Lemma for the wirelength computation.

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