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Odd Graceful Labelling of the Union of Cycle and Lobsters

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Abstract— Odd graceful labeling is one of the major evolving research areas in the field of graph labeling. It is defined as for any graph *G* with q edges if there is an injection *f* from *V(G)* to *{0, 1, 2, …, (2q-1)}* such that, when each edge *xy* is assigned the label $|f(x) - f(y)|$,so that the edge labels are *{1, 3, 5, …, (2q-1)}* then the graph G is said to be odd graceful. Graph labeling has a vast range of real life applications which has provided major contributions in the development of new technologies. In this paper we have investigated and proved that the graph *G* which is obtained by joining *m* isomorphic copies of lobster graph to each vertex of the cycle C_m admits odd graceful labeling.

Keywords— Graceful labeling**,** Odd graceful labeling, Cycle, Lobster.

I. Introduction

In Graph theory, graph labeling technique was first introduced in the mid *90's*. Most of the graph labeling methods has its origin from graceful labeling which was first introduced by Rosa*[9]* in the year *1967* and she called it as β- valuation. Later Golomb *[4]* called this β- valuation as graceful labeling in *1972*. Odd graceful labeling was introduced by Gananjothi*[3]* in the year *1991*. Assigning values to the vertices subject to certain conditions is known as Graph labeling. Gananjothi*[3]* has proved that the graph of path P_n and cycle C_n is odd graceful if and only if n is even. She has stated a very famous conjecture that All trees are odd graceful and also proved this for all trees with order up to *10*. In *2009* Barrientos*[1]* has verified this conjecture for order up to *12*. He also proved that every forest whose components are caterpillars (a caterpillar is a tree with the property that the removal of its end points leaves a path) are odd graceful. In *2010* Moussa*[6][7]* proved that the graph $C_m \cup P_n$ is odd graceful if m is even and provided an algorithm. . In *2009* Moussa and Badr*[8]* proved that $C_m \Theta P_n$ is odd graceful if and only if m is even. In *2002* David Morgan*[5]* proved that all lobsters (a lobsters is a tree with the property that the removal of the endpoints leaves a caterpillar) with perfect matching are odd graceful. In *2012* Zhou, Yao, Chen and Tao*[10]* proved that every lobster is odd graceful. For more results on odd graceful labeling, refer to dynamic survey by Gallian*[2]*.

Labeling has a vast range of application in communication network, optimal circuit layouts, cryptography, and traffic control systems.
In this paper we have proved that the graph obtained by joining each vertex of the cycle C_m of even order with *m* isomorphic copies of lobster admits odd graceful labeling.

II. Results and Discussion

In this section we prove that the graph *G* obtained by attaching each vertex of the cycle C_m with the *m* isomorphic copies of lobster graph is odd graceful where $m \equiv 0 (mod\ 4)$.

Theorem:
The graph *G* obtained by joining each vertex C_m with m copies of lobster graph is odd graceful where $m \equiv 0 (mod\ 4)$.
Proof:
Let G be a graph obtained by joining each vertex of cycle C_m with the lobster. Let $|V(G)| = p$ and $|E(G)| = q$.

The vertices in the cycle C_m are denoted as $u_1^1, u_1^2, u_1^3, \ldots, u_1^m$ in the clockwise direction. Consider a isomorphic m copies of lobster and attach it to each vertex of the cycle. The first copy of the part of the lobster attached to vertex u_1^1 of cycle C_m is denoted as $u_1^1, u_2^1, \ldots, u_n^1$. The second copy of the path of the lobster attached to vertex u_1^2 of cycle C_m is denoted as $u_1^2, u_2^2, \ldots, u_n^2$ continuing the same process the i^{th} copy of the path of the lobster attached with the vertex u_1^m of cycle C_m is denoted as $u_1^m, u_2^m, \ldots, u_n^m$.

The vertices in the first level of lobster attached with the odd vertices of the path of the cycle u_1^1 are denoted as $a_1^1, a_2^1, \ldots, a_r^1, a_{r+1}^1, a_{r+2}^1, \ldots, a_{2r}^1, a_{2r+1}^1, \ldots, a_{3r}^1, \ldots, a_{[(n/2)-1]+r+1}^1, a_{[(n/2)-1]+r+2}^1, \ldots, a_{(n/2)r}^1$. Vertices in the first level of lobster attached with the odd vertices of the path of the cycle u_1^2 are denoted as $a_{(n/2)r+1}^2, a_{(n/2)r+2}^2, \ldots, a_{(n/2)r+r}^2, a_{(n/2)r+r+1}^2, a_{(n/2)r+r+2}^2, \ldots, a_{(n/2)r+2r}^2, \ldots, a_{nr-r+1}^2, a_{nr-r+2}^2, \ldots, a_{nr}^2$. Vertices in the first level of lobster attached with the odd vertices of the path of the cycle u_1^3 are denoted as $a_{nr+1}^3, a_{nr+2}^3, \ldots, a_{nr+r}^3, a_{nr+r+1}^3, a_{nr+r+2}^3, \ldots, a_{nr+2r}^3, \ldots, a_{nr+[(n/2)-1]r+1}^3, a_{nr+[(n/2)-1]r+2}^3, \ldots, a_{3nr/2}^3$. Continuing the same process for the vertices in the first level of lobster attached with the odd vertices of the path of the cycle u_1^m are denoted as $a_{(m-1)nr/2+1}^m, a_{(m-1)nr/2+2}^m, \ldots, a_{(m-1)nr/2+r}^m, a_{(m-1)nr/2+r+1}^m, a_{(m-1)nr/2+r+2}^m, \ldots, a_{(m-1)nr/2+2r}^m, \ldots, a_{(mnr/2)-r+1}^m, a_{(mnr/2)-r+2}^m, \ldots, a_{mnr/2}^m$.

The vertices in the first level of lobster attached with the odd vertices of the path of the cycle u_1^1 are denoted as $b_1^1, b_2^1, \ldots, b_r^1, b_{r+1}^1, b_{r+2}^1, \ldots, b_{2r}^1, b_{2r+1}^1, \ldots, b_{3r}^1, \ldots, b_{[(n/2)-1]+r+1}^1, b_{[(n/2)-1]+r+2}^1, \ldots, b_{(n/2)r}^1$. Vertices in the first level of lobster attached with the odd vertices of the path of the cycle u_1^2 are denoted as $b_{(n/2)r+1}^2, b_{(n/2)r+2}^2, \ldots, b_{(n/2)r+r}^2, b_{(n/2)r+r+1}^2, b_{(n/2)r+r+2}^2, \ldots, b_{(n/2)r+2r}^2, \ldots, b_{nr-r+1}^2, b_{nr-r+2}^2, \ldots, b_{nr}^2$. Vertices in the first level of lobster attached with the odd vertices of the path of the cycle u_1^3 are denoted as $b_{nr+1}^3, b_{nr+2}^3, \ldots, b_{nr+r}^3, b_{nr+r+1}^3, b_{nr+r+2}^3, \ldots, b_{nr+2r}^3, \ldots, b_{nr+[(n/2)-1]r+1}^3, b_{nr+[(n/2)-1]r+2}^3, \ldots, b_{3nr/2}^3$. Continuing the same process for the vertices in the first level of lobster attached with the odd vertices of the path of the cycle u_1^m are denoted as $b_{(m-1)nr/2+1}^m, b_{(m-1)nr/2+2}^m, \ldots, b_{(m-1)nr/2+r}^m, b_{(m-1)nr/2+r+1}^m, b_{(m-1)nr/2+r+2}^m, \ldots, b_{(m-1)nr/2+2r}^m, \ldots, b_{(mnr/2)-r+1}^m, b_{(mnr/2)-r+2}^m, \ldots, a_{mnr/2}^m$.

The vertices in the second level of lobster attached with the odd vertices of the path u_n^m is denoted as $a_1^2, a_2^2, \ldots, a_s^2, a_{s+1}^2, a_{s+2}^2, \ldots, a_{2s}^2, a_{2s+1}^2, \ldots, a_{3s}^2, \ldots, a_{(n-1)s+1}^2, \ldots, a_{rs}^2, a_{rs+1}^2, \ldots, a_{2rs}^2, a_{2rs+1}^2, \ldots, a_{3rs}^2$ and so on the last vertex of the level two in this path will be $a_{(n/2)rs}^2$. Vertices in the second level of lobster attached with the odd vertices of the path u_{n-2}^m is denoted as $a_{[(n/2)rs]+1}^2, a_{[(n/2)rs]+2}^2, \ldots, a_{[(n/2)rs]+s}^2, a_{[(n/2)rs]+s+1}^2, a_{[(n/2)rs]+2}^2, \ldots, a_{[(n/2)rs]+2s}^2, a_{[(n/2)rs]+2s+1}^2, \ldots, a_{[(n/2)rs]+3s}^2, \ldots, a_{[(n/2)rs]+rs}^2, a_{[(n/2)rs]+rs+1}^2, \ldots, a_{[(n/2)rs]+2rs}^2, a_{[(n/2)rs]+2rs+1}^2, \ldots, a_{[(n/2)rs]+3rs}^2$ and so on the last vertex of the level two in this path will be a_{nrs}^2. The last vertex in the level two of the next path will be $a_{3(n/2)rs}^2$. Continuing the same way the last vertex in the second level of lobster attached with the odd vertices of the path will be $a_{m(n/2)rs}^2$.

The vertex in the level *2* attached with the odd vertex of the path u_{n-1}^m is denoted as $b_1^2, b_2^2, \ldots, b_s^2, b_{s+1}^2, b_{s+2}^2, \ldots, b_{2s}^2, b_{2s+1}^2, \ldots, b_{3s}^2, \ldots, b_{(n-1)s+1}^2, \ldots, b_{rs}^2, b_{rs+1}^2, \ldots, b_{2rs}^2, b_{2rs+1}^2, \ldots, b_{3rs}^2$ and so on the last vertex of the level two in this path will be $b_{(n/2)rs}^2$. The vertex in the level *2* attached with the even vertex of the path u_{n-3}^m is denoted as $b_{[(n/2)rs]+1}^2, b_{[(n/2)rs]+2}^2, \ldots, b_{[(n/2)rs]+s}^2, b_{[(n/2)rs]+s+1}^2, b_{[(n/2)rs]+2}^2, \ldots, b_{[(n/2)rs]+2s}^2, b_{[(n/2)rs]+2s+1}^2, \ldots, b_{[(n/2)rs]+3s}^2, \ldots, b_{[(n/2)rs]+rs}^2, b_{[(n/2)rs]+rs+1}^2, \ldots, b_{[(n/2)rs]+2rs}^2, b_{[(n/2)rs]+2rs+1}^2, \ldots, b_{[(n/2)rs]+3rs}^2$ and so on the last vertex of the level two in this path will be b_{nrs}^2. The last vertex in the level two of the next path will be $b_{3(n/2)rs}^2$. Continuing the same way the last vertex in the level *2* in the last path will be $b_{m(n/2)rs}^2$.

Here, N = (the number of edges of level one + the number edges of the path) in a single copy, r is the number of first level vertives attached in one vertex of the path of the lobster, s is the number of second level vertices attached in one first level vertex of the lobster, k denotes the m^{th} vertex in the cycle, i denotes the n^{th} path of the lobster, g denotes the s^{th} vertex of the lobster and h denotes the h^{th} vertex of the lobster.

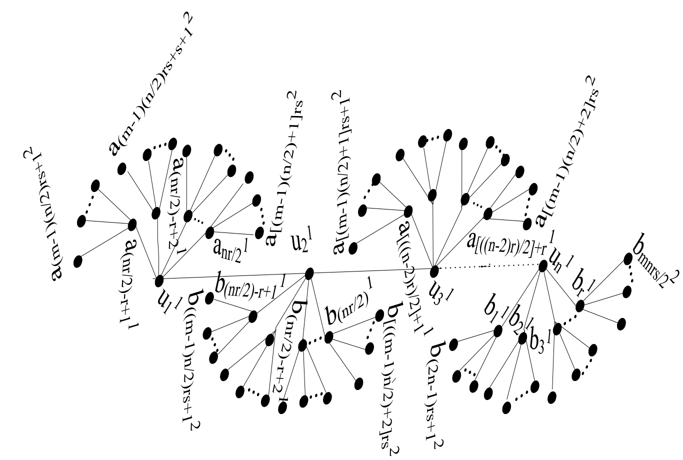

Figure 1: The vertex in the first copy of the lobster

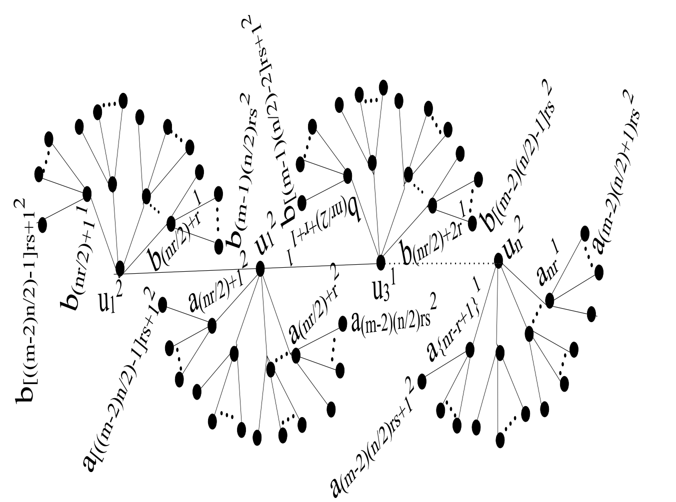

Figure 2: The vertex in the second copy of the lobster

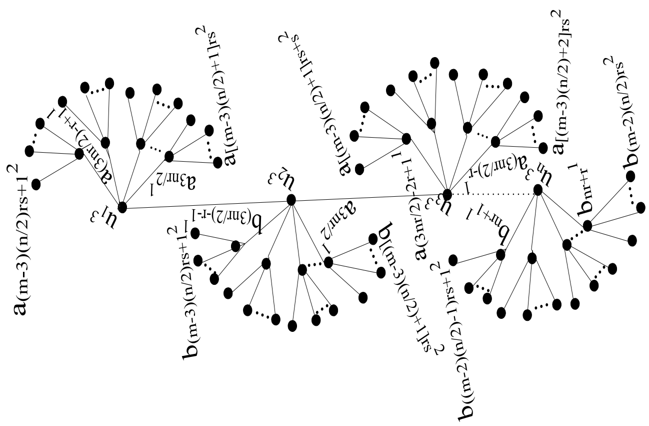

Figure 3: The vertex in the third copy of the lobster

And so on the vertices in the last copy of the lobster is given as:

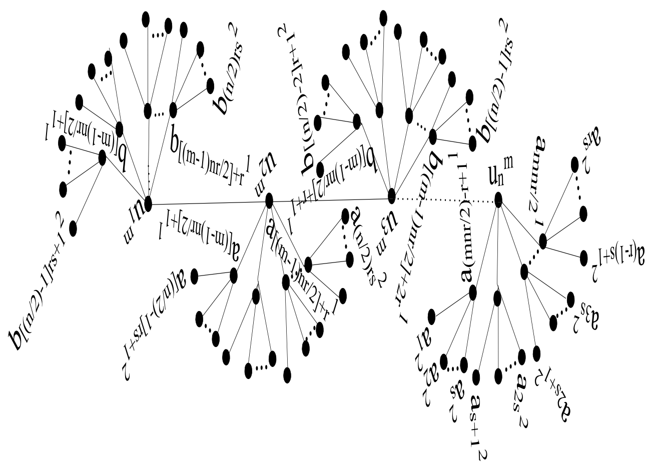

Figure 4: Vertices in the last copy of the lobster

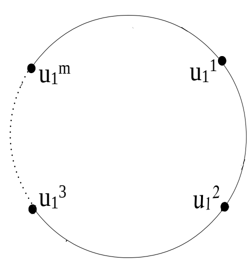

Figure 5: The vertices in the cycle C_m

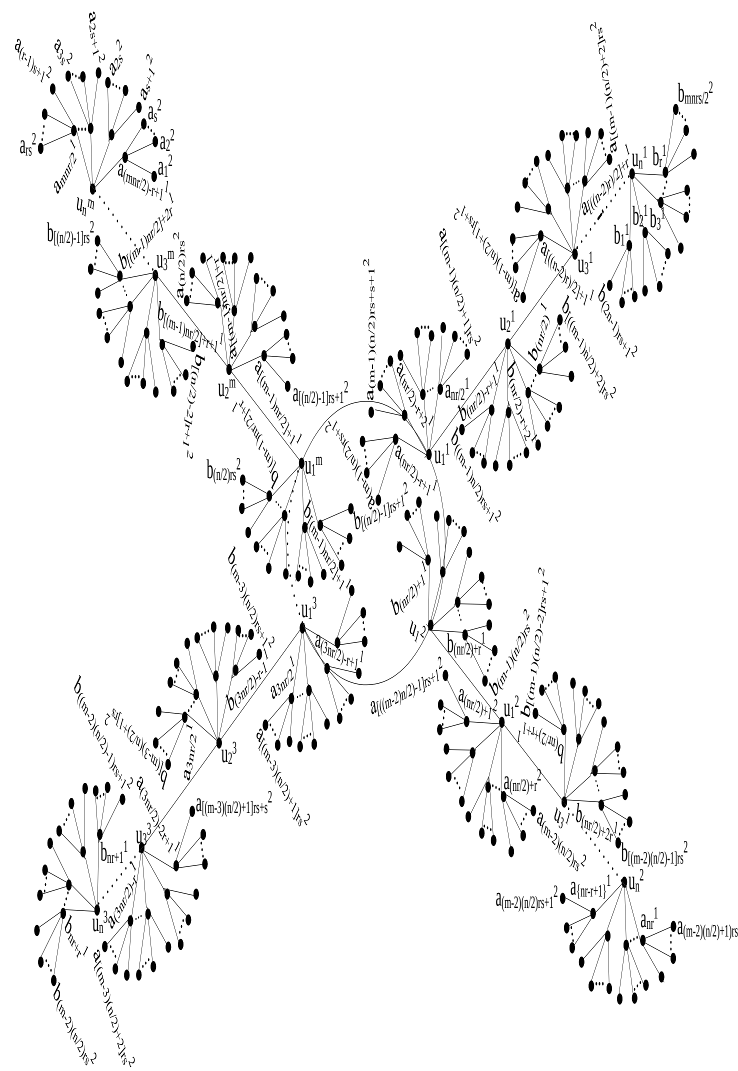

Figure 6: The graph G of Cycle with lobster

The vertex label of the cycle C_m is given as:

$$f(u_1^{2k-1}) = \begin{cases} (2q-1)-(N-1)-2(N+1)(k-1), & 1 \le k \le m/4 \\ (2q-1)-(N-1)-2(N+1)(k-1)-2, & \frac{m}{4}+1 \le i \le m/2 \end{cases}$$

$f(u_1^{2k}) = (N+1)+2(N+1)(k-1), \quad 1 < k < m/2$

The vertices in the path of the lobster, m is even are labeled based on the parameter *'N'*, *'n'* and *'r'* as follows:

$$f(u_{2i-1}^{2k-1}) = \begin{cases} (2q-1)-(N-1)-2(N+1)(k-1)+2(r+1)(i-1), & 1 \le i \le \frac{n+1}{2}, 1 \le k \le \frac{m}{4} \\ (2q-1)-(N-1)-2(N+1)(k-1)+2(r+1)(i-1)-2, & 1 \le i \le \frac{n+1}{2}, \frac{m}{4}+1 \le k \le \frac{m}{2} \end{cases}$$

$f(u_{2i-1}^{2k}) = (N+1)+2(N+1)(k-1)+2(r+1)(i-1),$
$1 \le i \le (n+1)/2,\ 1 \le k \le m/2$

$f(u_{2i}^{2k-1}) = (N+1)+2(N+1)(k-1)-2(r+1)(i-1),$
$1 \le i \le (n-1)/2,\ 1 \le k \le m/2$

$$f(u_{2i}^{2k})= \begin{cases} (2q-1)-2(N+1)(k-1)-2(r+1)(i-1)-2(N-3)+2(r-2), & 1 \le i \le (n+1)/2,\ 1 \le k \le m/4 \\ (2q-1)-2(N+1)(k-1)-2(r+1)(i-1)-2(N-3)+2(r-2)-2, & 1 \le i \le (n+1)/2,\ 1 \le k \le m/2 \end{cases}$$

The vertex labels for level *1* of Lobster at u_i is defined as follows,

$f(a_g^{1}) = 2i, \quad 1 \le g \le r$
$f(a_{tr+g}^{1}) = f(a_{tr}^{1})+4+(2g-2), \quad 1 \le g \le r,\ 1 \le t \le mn/2$
$f(b_g^{1}) = (2q-1)-2(g-1), \quad 1 \le g \le r$
$f(b_{tr+g}^{1}) = f(b_{tr}^{1})-4-2(g-1), \quad 1 \le g \le r, 1 \le t \le mn/4$
$f(b_{mnr/4+g}^{1}) = f(b_{mn/4})-6-2(g-1), \quad 1 \le g \le r$

The vertex labels for level *2* of Lobster at a_g is defined as follows, where *a* is even.

$f(a_h^{2}) = (2q-1)-2(N+1)(k-1)+2(r+1)(i-1)-(N-3)+2(j-1)-2-2(h-1), \quad i = n/2, k = m/2$
$f(a_{gs+h}^{2}) = f(a_{gs}^{2})-4-2(h-1), \quad 1 \le g \le r-1$
$f(a_{rs+h}^{2}) = f(a_{rs}^{2})-2(rs+3)-2(h-1), \quad 1 \le h \le s$
$f(a_{trs+h}^{2}) = f(a_{trs}^{2})-2(rs+3)-2(h-1), \quad 1 \le t \le mn/2$

The vertex labels for level *2* of Lobster at b_g is defined as follows, where *b* is odd.

$f(b_h^{2}) = f(b_{mnr/2}^{1})+2(rs)+4+2(h-1), \quad 1 \le h \le s$
$f(b_{gs+h}^{2}) = f(b_{gs}^{2})+4+2(h-1), \quad 1 \le h \le s,\ 1 \le g \le r-1$
$f(b_{trs+h}^{2}) = f(b_{trs}^{2})+2(rs+3)+2(h-1),$
$1 \le h \le s,\ 1 \le t \le mn/4$
$f(b_{s(tr+g)+h}^{2}) = f(b_{s(tr+g)}^{2})+2(h-1)+4, \quad 1 \le h \le s,\ 1 \le g \le r-1$
$f(b_{smnr/4+h}^{2}) = f(b_{mnrs/4}^{2})+2(rs+3)+2(h-1)+2, \quad 1 \le g \le r$
$f(b_{trs+h}^{2}) = f(b_{trs}^{2})+2(rs+3)+2(h-1), \quad mn/4+s \le t \le mn/2$

From the above equation it is clearly seen that the vertex labels are distinct and edge get the odd labels from *1* to *2q-1*.Thus the graph *G* is odd graceful. Illustration of the graph G is given in figure 7.

Illustration:

When $m=4, n=4, r=4, s=3, p=272, q=272$

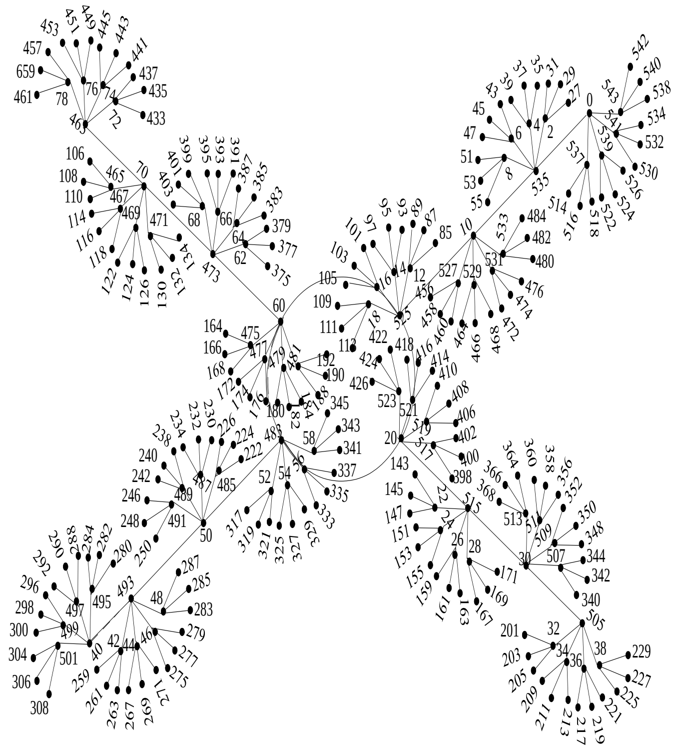

Figure 7: Illustration of the graph G

III. CONCLUSION AND FUTURE SCOPE

Thus we have proved that the graph *G* obtained by attaching each vertex of the cycle C_m with the *m* isomorphic copies of lobster graph is odd graceful where $m \equiv 0 (mod\ 4)$.

Further we are intended to prove that the graph *G* obtained by attaching each vertex of the cycle C_m with the m arbitrary copies of lobster graph is odd graceful where $m \equiv 0 (mod\ 4)$.

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